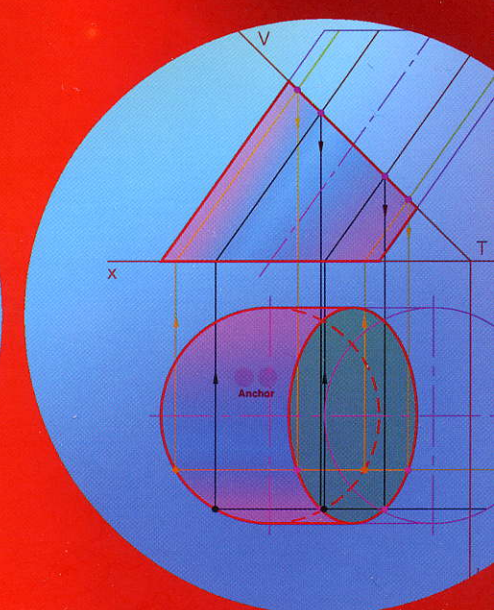
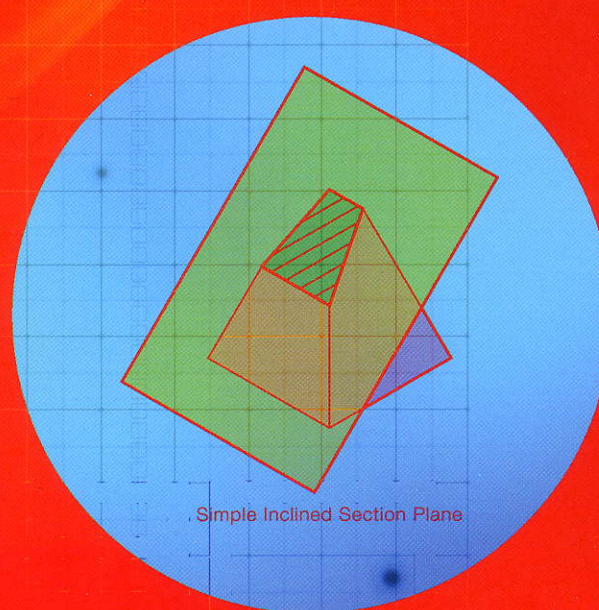
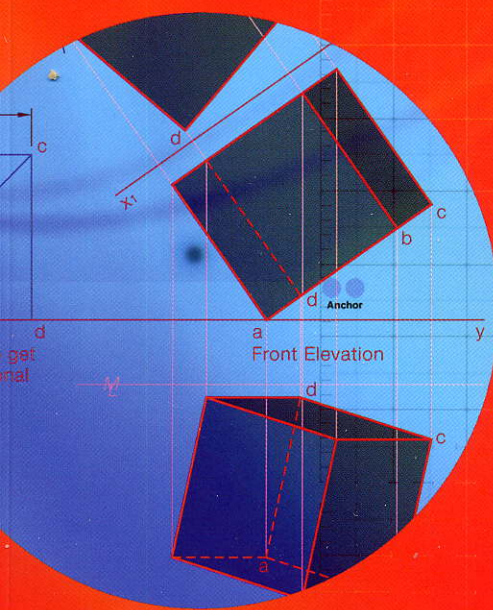


GRAPHICS IN DESIGN & COMMUNICATION

1

PLANE AND DESCRIPTIVE GEOMETRY



DAVID ANDERSON

8

Conic Sections

SYLLABUS OUTLINE

Areas to be studied:

- Terminology for conics. • The ellipse, parabola and hyperbola as sections of a right cone.
- Understanding of focal points, focal sphere, directrix and eccentricity in the context of conic sections.
 - *Derivation of focal points, directrix and eccentricity using the focal sphere and solid cone.*
- Construction of conic curves as geometric loci. • Geometric properties common to the conic curves.
 - Tangents to conics.
 - *Construction of hyperbolae from focal points and transverse axis.*

Learning outcomes

Students should be able to:

Higher and Ordinary levels

- Understand the terms used in the study of conics, e.g. chord, focal chord, directrix, vertex, ordinate, tangent, normal, major and minor axes/auxiliary circles, eccentricity, transverse axis.
- Construct ellipse, parabola, hyperbola as true sections of solid cone.
- Construct the conic sections, the ellipse, parabola and hyperbola, as plane loci from given data relating to eccentricity, foci, vertices, directrices and given points on the curve.
- Construct ellipse, parabola and hyperbola in a rectangle given principal vertex(s).
- Construct tangents to the conic sections from points on the curve.

Higher level only

- *Understand the terms used in the study of conics, double ordinate, latus rectum, focal sphere etc.*
- *Construct ellipse, parabola, hyperbola as true sections of solid cone and derive directrices, foci, vertices and eccentricity of these curves.*
- *Construct tangents to the conic sections from points outside the curve.*
- *Construct a double hyperbola given the foci and a point on the curve, or given the length of the transverse axis and the foci.*
- *Determine the centre of curvature and evolute for conic sections.*

Hyperbola as a Section of a Cone

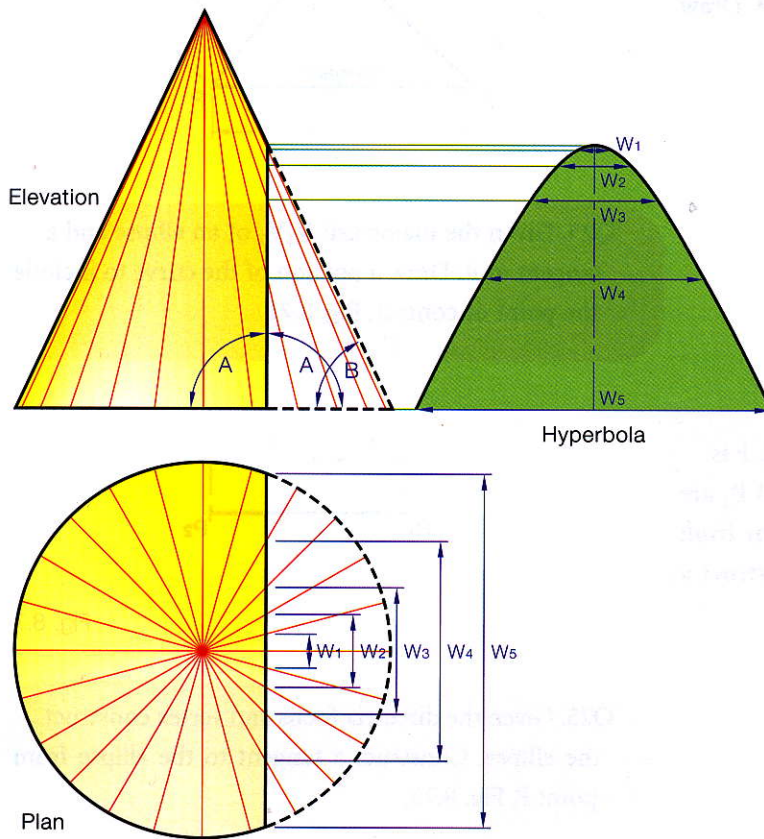


Fig. 8.77

If a cone is cut by a cutting plane such that either angle at the base (angle A) is equal to 90° , or is between 90° and the angle of the side of the cone (angle B), then a hyperbola is produced. The cutting plane only cuts one side of the cone no matter how far the cone and plane are extended.

CONSTRUCTION

Fig. 8.77

The construction is the same as that for the parabola and ellipse.

- (1) Draw the plan, elevation, elements and cutting plane.
- (2) The auxiliary view is projected perpendicularly from the cutting plane in elevation.
- (3) Points where the cutting plane crosses elements are projected out.
- (4) Widths are taken from the plan as shown.

Focal Sphere

The hyperbola has one focal sphere. The position and size of the sphere is such that it is tangential to both the sides of the cone and the cutting plane, i.e. it fits neatly into the top of the cone and also touches the cutting plane at one spot, the focus. The directrix is found by finding the intersection between the plane through the sphere containing all the points of contact between the sphere and the cone and the cutting plane.

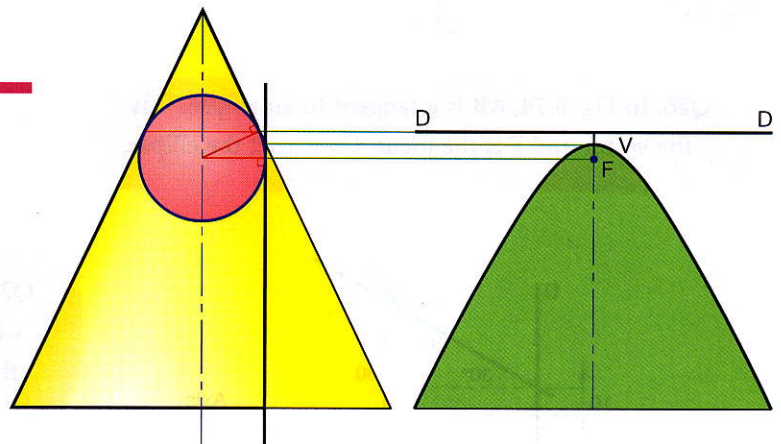


Fig. 8.78

Terminology

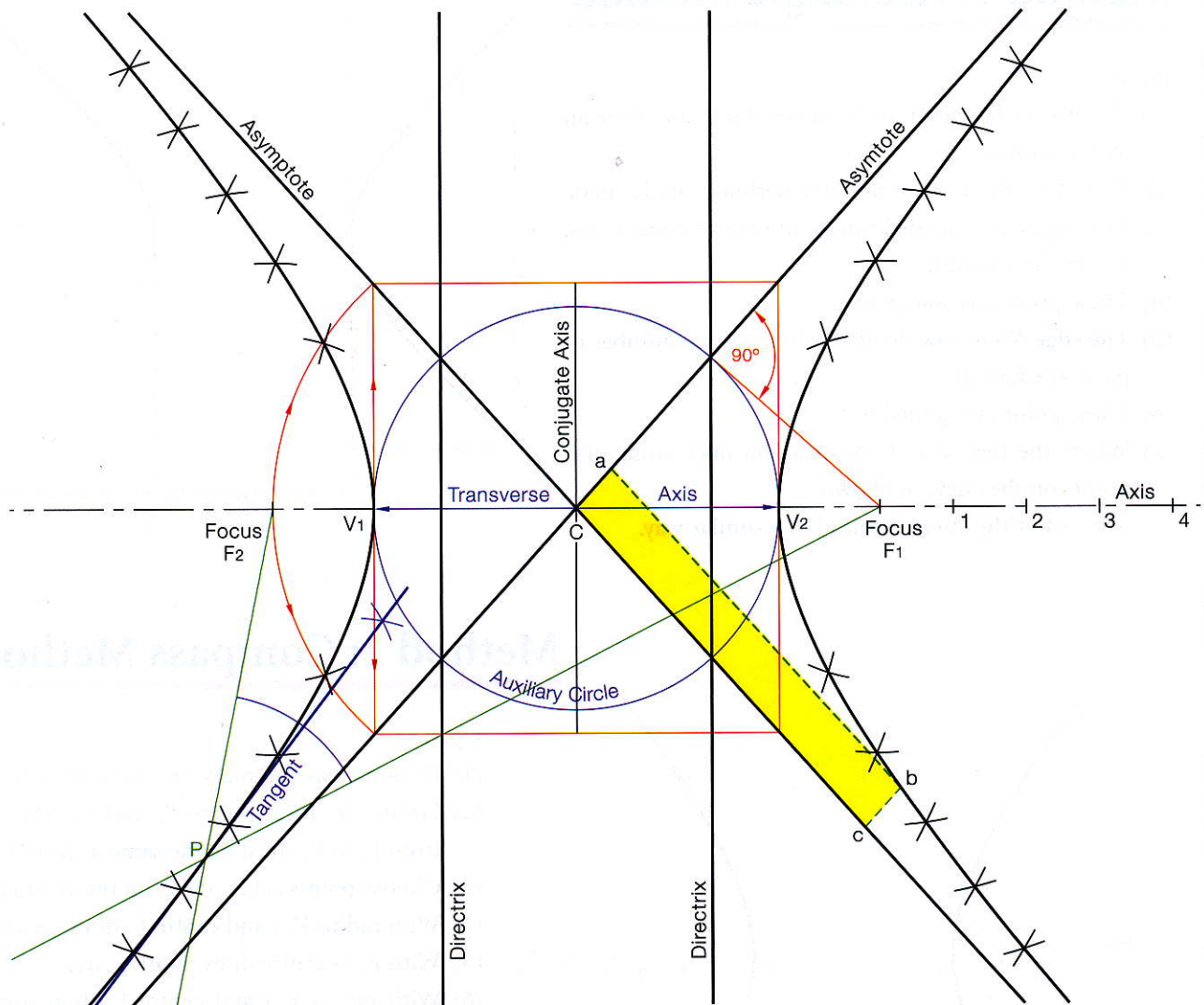


Fig. 8.79

Distance from Focus₂ to P – Distance from Focus₁ to P = Transverse Axis

$F_2P - F_1P = \text{Transverse Axis}$

$F_2P - F_1P = V_1V_2$

Transverse Axis – The part of the axis between the vertices V_1 to $V_2 = \text{Transverse Axis}$.

Auxiliary Circle – The circle passing through both vertices and having its centre on the axis at C is called the auxiliary circle.

Asymptotes – These are the outer limits of the cones. They are straight lines which cross at C. The hyperbola curve will get closer and closer to the asymptote but will never touch it.

Conjugate Axis – A perpendicular to the axis through the centre C is called the conjugate axis.

Rectangle abcC – Select any point b on the curve. From b draw ba and bc parallel to the asymptotes. The resulting parallelogram has an area which is constant no matter where on the curve point b is selected.

Three Methods of Constructing a Hyperbola

Method 1: Rectangle Method

Fig. 8.80

- (1) The two rectangles must be of equal size and share an axis as shown.
- (2) V_1 and V_2 , the vertices of the hyperbolas, are located.
- (3) The edge CB is divided into a number of equal parts, five in this example.
- (4) These points are joined to V_2 .
- (5) The edge AB is now divided into the same number of parts as edge CB.
- (6) These points are joined to V_1 .
- (7) Where the two sets of construction lines cross, plot points on the curve as shown.
- (8) The rest of the curve is found in a similar way.

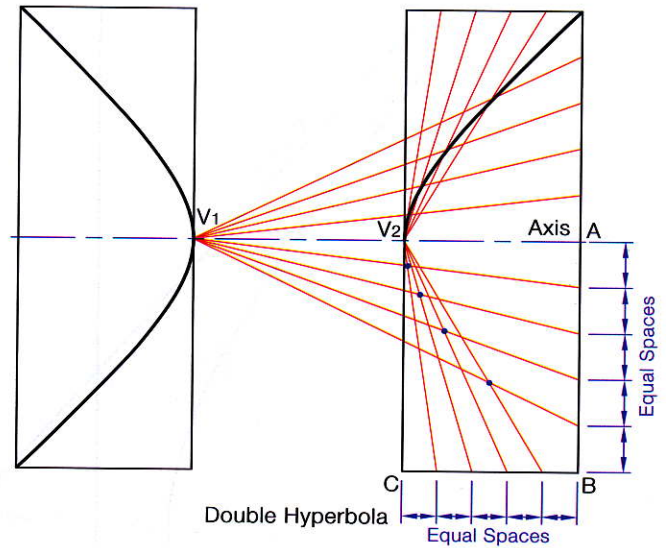


Fig. 8.80

Method 2: Compass Method

Fig. 8.81

- (1) Draw an axis and mark the two vertices V_1 and V_2 .
- (2) Locate the focal points F_1 and F_2 . The distance from F_1 to V_1 must be the same as from F_2 to V_2 .
- (3) Choose points a, b, c, d etc. on the axis beyond F_2 .
- (4) With radius V_1a and centre F_1 draw an arc.
- (5) With F_2 as centre draw another arc.
- (6) With radius V_2a and centre F_2 draw arcs above and below the axis to cut the previous arc giving two points on one branch of the curve.
- (7) Using the same radius repeat this process using F_1 as centre giving two points on the second branch of the curve.
- (8) Repeat this process using points b, c, d etc. plotting the path of the curve.

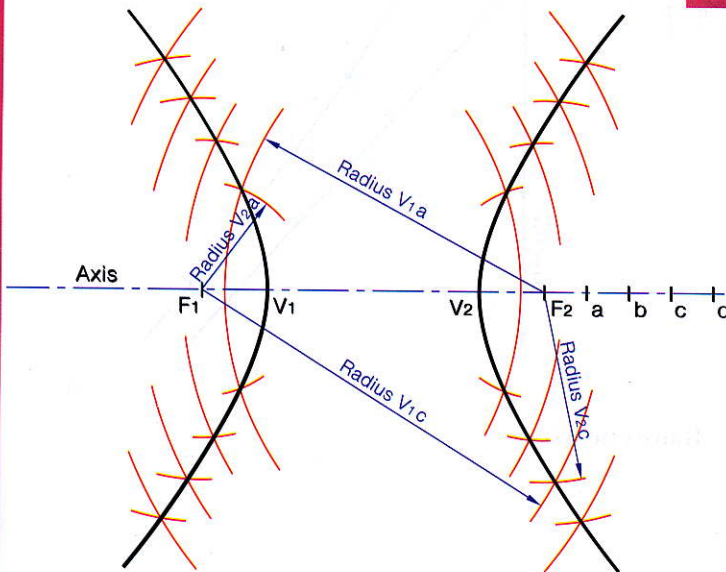


Fig. 8.81

Method 3: Eccentricity Method

Fig. 8.82

The eccentricity for a hyperbola is always **greater than one**, e.g. 7/4, 12/5, 1.6 etc.

The eccentricity line for a hyperbola will always make an angle of greater than 45° with the axis.

Eccentricity is a ratio between two distances

$$= \frac{\text{Distance from focus to a point on the curve}}{\text{Distance from the same point to directrix}}$$

$$= \frac{F \text{ to } P}{P \text{ to } DD}$$

The ratio is a constant for a particular curve, no matter where on the curve the point P is chosen.

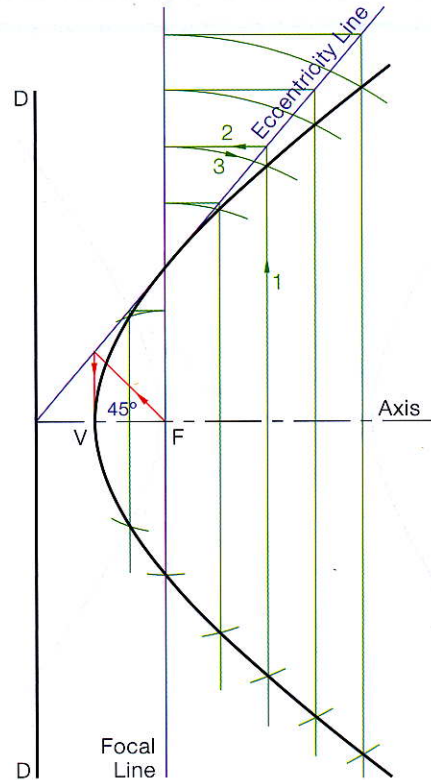


Fig. 8.82

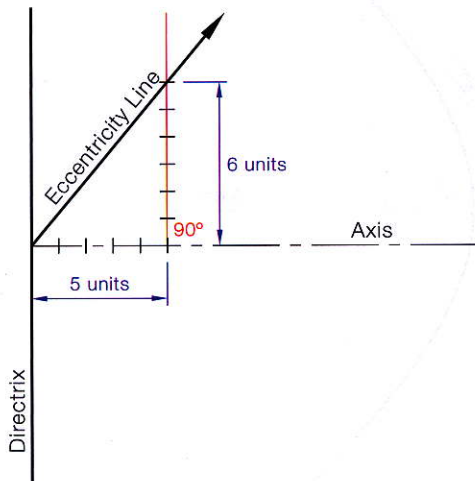


Fig. 8.83

CONSTRUCTION

Given directrix focus and eccentricity of 1.2.

The advantage of the eccentricity method is that once the eccentricity line is set up correctly the process of construction is the same for all three conics.

- (1) Draw the directrix, axis and focus.
- (2) Eccentricity of 1.2 can be written as a fraction 6/5.
- (3) Measure out from the directrix five units, e.g. $5 \times (5 \text{ mm units}) = 25 \text{ mm}$, and mark a point on the axis.
- (4) Draw a perpendicular to the axis at this point and measure up from the axis six units, e.g. $6 \times (5 \text{ mm units}) = 30 \text{ mm}$.
- (5) The eccentricity line can now be drawn.

Note: The bottom portion of the eccentricity fraction is measured out along the axis and the top is measured up on a perpendicular.

- (6) A line drawn up from the focus, at 45° to the axis, to hit the eccentricity line and dropped to the axis finds the vertex.
- (7) Construction for the curve is the same as for the ellipse and parabola.

Tangent to a Hyperbola from a Point on the Curve

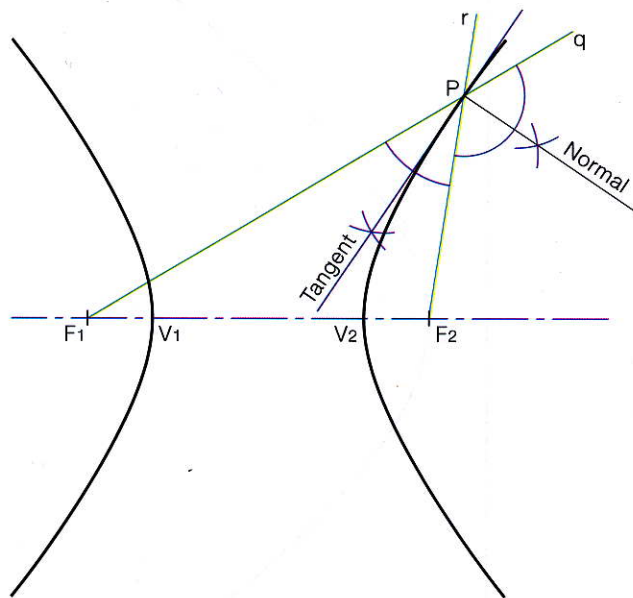


Fig. 8.84

Method 1

Fig. 8.84

- (1) Join F_1 to P and extend to q .
- (2) Join F_2 to P and extend to r .
- (3) The line that bisects angle F_1PF_2 or angle qPr will be the tangent.
- (4) The normal is perpendicular to the tangent and can be found by bisecting the angle F_2Pq .

Note: The similarity to the method used for the parabola, Fig. 8.10, and the ellipse, Fig. 8.44.

Method 2

Fig. 8.85

This method works for all three conics and has already been shown in Fig. 8.8 for the parabola and Fig. 8.43 for the ellipse.

- (1) Join P back to the focus.
- (2) At the focus draw a perpendicular to PF to hit the directrix at q .
- (3) Point q is a point on the tangent. Join P to q , forming the tangent.

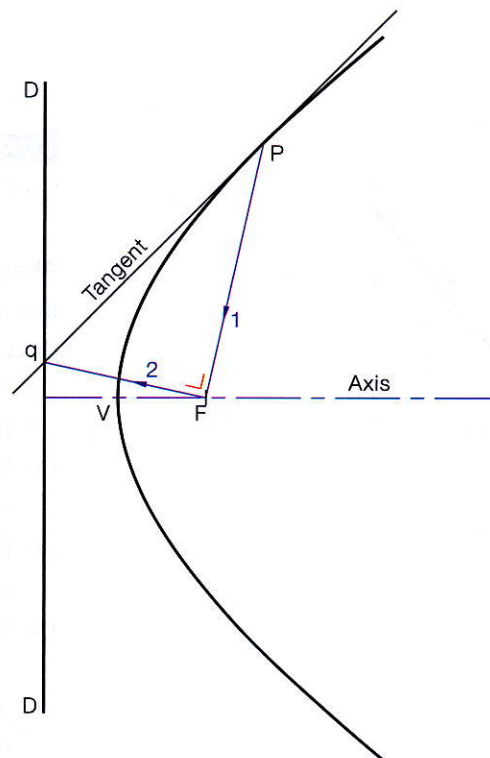


Fig. 8.85

Tangent to a Hyperbola from a Point P outside the Curve

Method 1

Fig. 8.86

- (1) Join P to F_2 and place a circle on this line having PF_2 as its diameter.
- (2) Draw the auxiliary circle for the double hyperbola, i.e. the circle which has V_1V_2 as its diameter.
- (3) The circles cross at q and r which will be points on the tangents.
- (4) Draw the tangent p to q extended.

Note: The similarity to construction for a parabola, Fig. 8.14, and construction for an ellipse, Fig. 8.46.

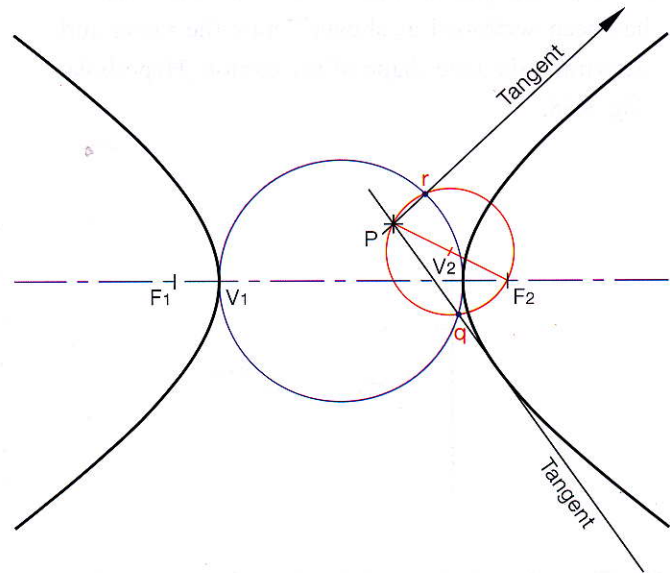


Fig. 8.86

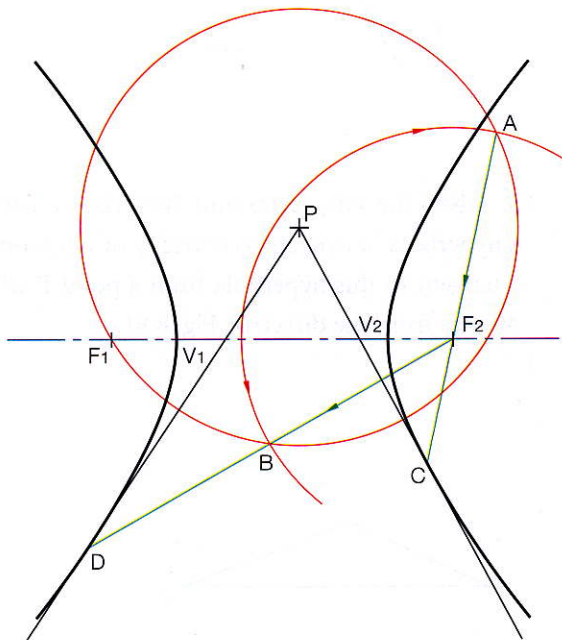


Fig. 8.87

Method 2

Fig. 8.87

- (1) Draw a circle with P as centre and PF_1 as radius.
- (2) With radius V_1V_2 (transverse axis) and F_2 as centre, scribe an arc to cut this circle at A and B.
- (3) Join A to F_2 and extend to C, join B to F_2 and extend to D.
- (4) Draw the tangents PC and PD.

Note: Similar to the construction for an ellipse, Fig. 8.47.

Activities

Q1. Given the plan and elevation of a cone which has been sectioned as shown. Draw the views and determine the true shape of the section. Hyperbola, Fig. 8.88.

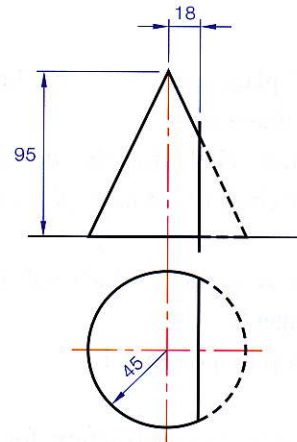
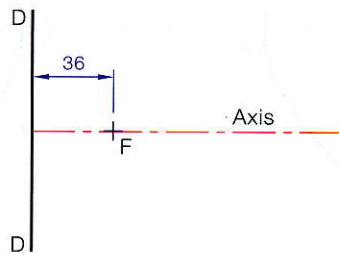


Fig. 8.88

Fig. 8.89



Q2. Given the directrix, axis, focus and eccentricity of $6/5$. Draw a portion of the hyperbola, Fig. 8.89.

Q3. Given the axis, focus, point P on the curve and eccentricity of 1.25. Draw a portion of the hyperbola Fig. 8.90.

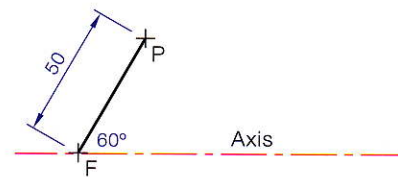


Fig. 8.90

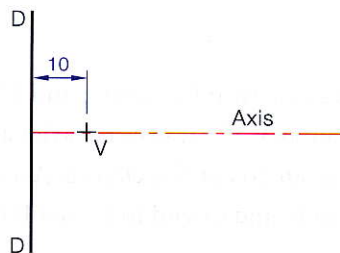


Fig. 8.91

Q4. Given the axis, vertex and directrix. Construct a hyperbola having an eccentricity of $7/5$. Construct a tangent to this hyperbola from a point P which is 40 mm from the directrix, Fig. 8.91.

Q5. Construct the triangle P_1P_2F where P_1 and P_2 are points on the curve of a hyperbola and F is its focus. The eccentricity is $10/7$, Fig. 8.92.

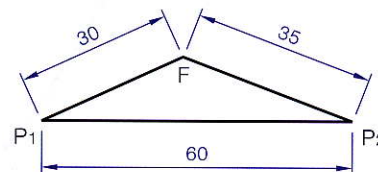


Fig. 8.92

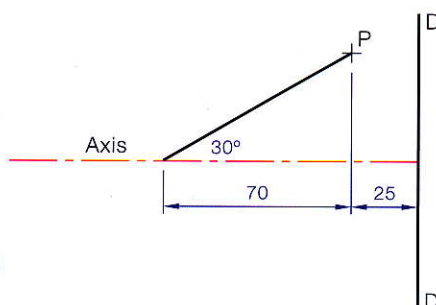


Fig. 8.93

Q6. Given the normal, axis, point of contact and directrix. Find the focus and construct a portion of the curve, Fig. 8.93.

Q7. Given the focus and vertex of a hyperbola having an eccentricity of 1.1. Draw a portion of the curve, Fig. 8.94.

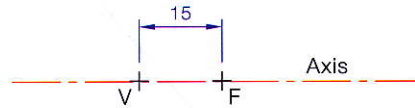


Fig. 8.94

Q8. Given the axis, directrix, tangent and point of contact P. Draw a portion of the hyperbola, Fig. 8.95.

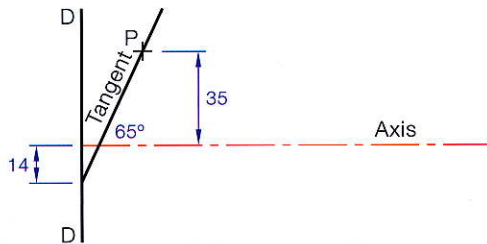


Fig. 8.95

Q9. Given the cone in Fig. 8.96 which is sectioned as shown. Construct the focal sphere and hence find the focus, vertex and directrix of the hyperbola. Draw the hyperbola.

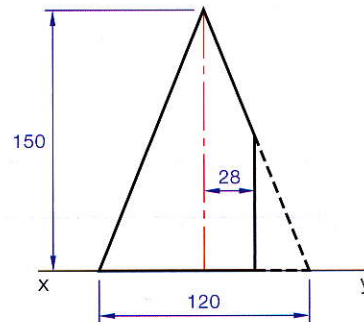


Fig. 8.96

Q10. Construct a double hyperbola in the rectangles shown, Fig. 8.97.

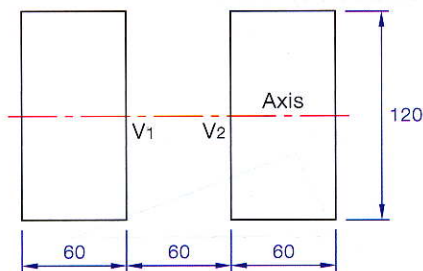


Fig. 8.97

Q11. Given the directrix, vertex, axis and focus. Draw the hyperbola. Construct a tangent to the curve from the given point P, Fig. 8.98.

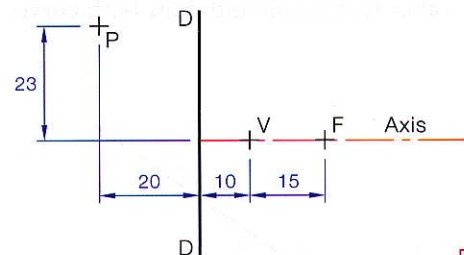


Fig. 8.98

Q12. Given the foci and vertices of a double hyperbola. Construct a portion of each curve. Construct a tangent to the curve from a point 50 mm from F_1 , Fig. 8.99.

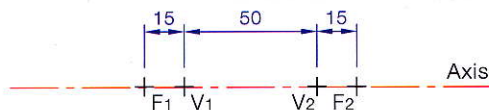


Fig. 8.99

Q13. Given the transverse axis of 80 mm and the distance between F and V of 20 mm. Construct a double hyperbola, directrices, asymptotes, auxiliary circle and conjugate diameter.

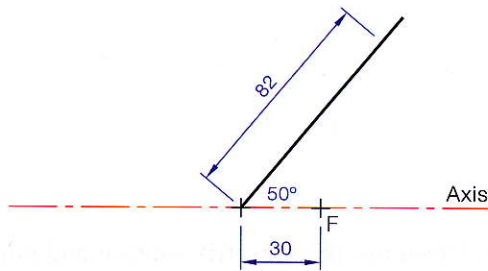


Fig. 8.100

Q15. Given the line P_1FP_2 as shown in Fig. 8.101. P_1 and P_2 are points on the curve of a hyperbola, F is the focus and the eccentricity is $7/5$. Draw the curve.

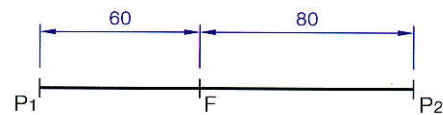


Fig. 8.101

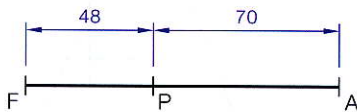


Fig. 8.102

Q16. Given the line FPA as shown in Fig. 8.102. If F is the focus, P is a point on the curve and A is a point on the directrix. Construct the hyperbola if the eccentricity is $6/5$.

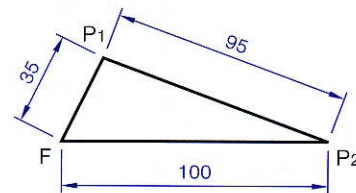


Fig. 8.103

Q17. Given the triangle FP_1P_2 , F is one of the focal points on a double hyperbola, P_1 is a point on one branch of the curve and P_2 is a point on the other branch. The transverse axis is 60 mm long. Find the other focal point and draw both curves, Fig. 8.103.

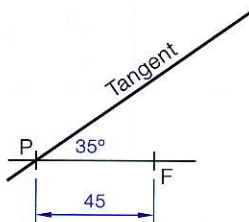


Fig. 8.104

Q18. Given a tangent, the point of contact P and the focal point of a double hyperbola. The transverse axis is 50 mm. Draw a portion of the double hyperbola, Fig. 8.104.

Q19. Given the two focal points of a double hyperbola and a point P on one branch of the curve. Draw the double curve, Fig. 8.105.

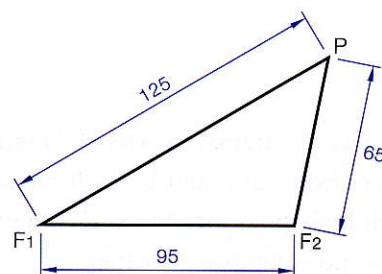


Fig. 8.105