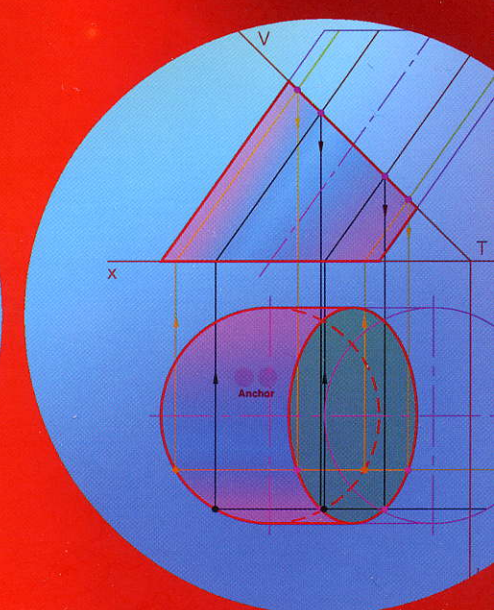
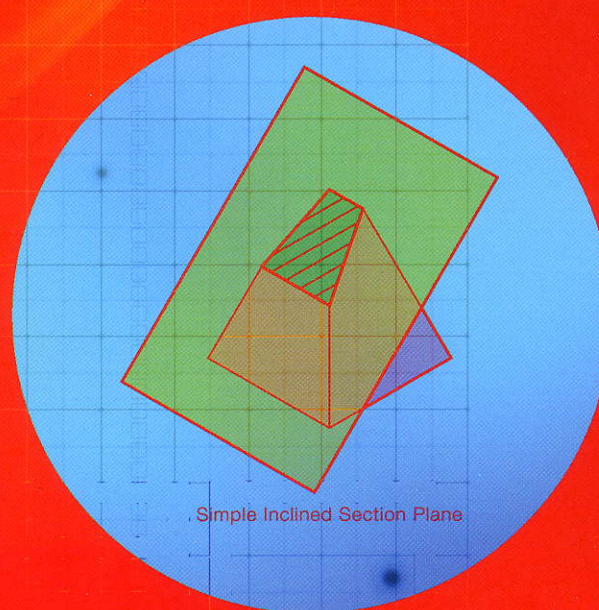
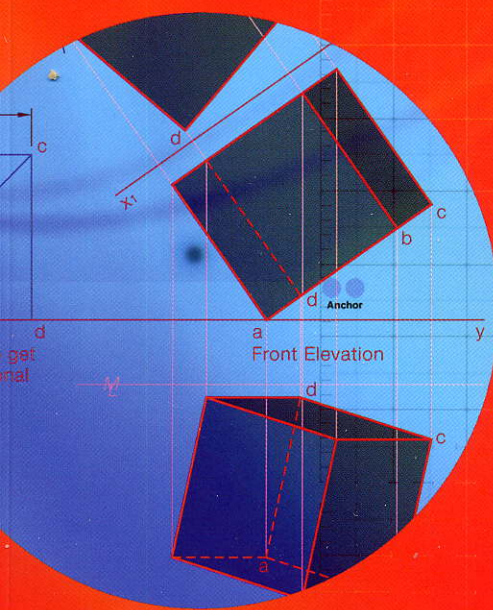


GRAPHICS IN DESIGN & COMMUNICATION

1

PLANE AND DESCRIPTIVE GEOMETRY



DAVID ANDERSON

8

Conic Sections

SYLLABUS OUTLINE

Areas to be studied:

- Terminology for conics. • The ellipse, parabola and hyperbola as sections of a right cone.
- Understanding of focal points, focal sphere, directrix and eccentricity in the context of conic sections.
 - *Derivation of focal points, directrix and eccentricity using the focal sphere and solid cone.*
- Construction of conic curves as geometric loci. • Geometric properties common to the conic curves.
 - Tangents to conics.
 - *Construction of hyperbolae from focal points and transverse axis.*

Learning outcomes

Students should be able to:

Higher and Ordinary levels

- Understand the terms used in the study of conics, e.g. chord, focal chord, directrix, vertex, ordinate, tangent, normal, major and minor axes/auxiliary circles, eccentricity, transverse axis.
- Construct ellipse, parabola, hyperbola as true sections of solid cone.
- Construct the conic sections, the ellipse, parabola and hyperbola, as plane loci from given data relating to eccentricity, foci, vertices, directrices and given points on the curve.
- Construct ellipse, parabola and hyperbola in a rectangle given principal vertex(s).
- Construct tangents to the conic sections from points on the curve.

Higher level only

- *Understand the terms used in the study of conics, double ordinate, latus rectum, focal sphere etc.*
- *Construct ellipse, parabola, hyperbola as true sections of solid cone and derive directrices, foci, vertices and eccentricity of these curves.*
- *Construct tangents to the conic sections from points outside the curve.*
- *Construct a double hyperbola given the foci and a point on the curve, or given the length of the transverse axis and the foci.*
- *Determine the centre of curvature and evolute for conic sections.*

Ellipse as a Section of a Cone

An ellipse is produced by a cutting plane which passes through both sides of a cone or will cut both sides of the cone when extended.

CONSTRUCTION

- (1) Draw the plane and elevation of a cone.
- (2) Divide the plan into sections using generators and project these onto the elevation.
- (3) In elevation, draw in the cutting plane so that it cuts both sides of the cone or will cut both sides of the cone if the plane and cone are extended.
- (4) The points where the generators are cut in elevation are projected down to the plan producing a curve as shown.
- (5) The true shape of this curve is an ellipse and is produced by projecting perpendicularly from the cutting plane. The widths are taken from the plan.

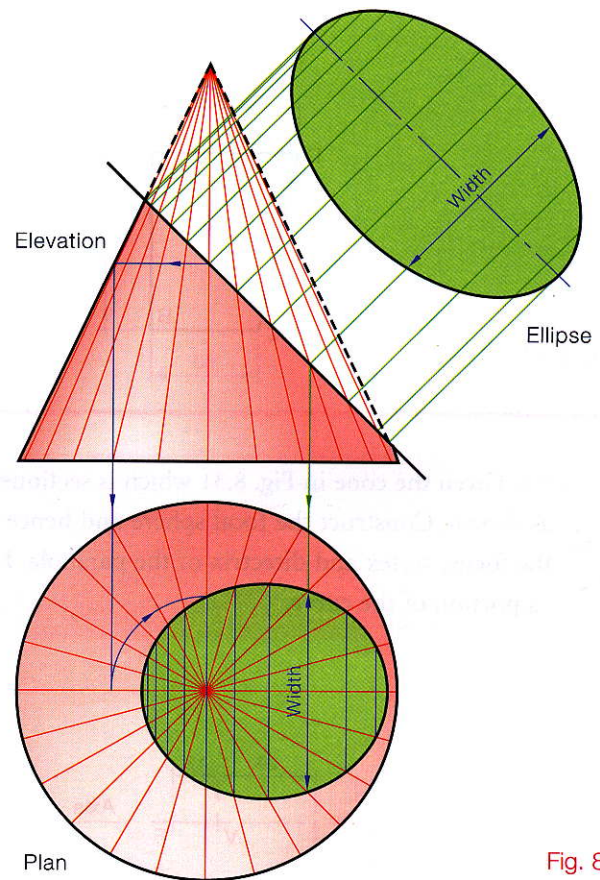


Fig. 8.34

Focal Spheres for an Ellipse

Similar to the construction for a parabola a sphere is inserted into the space between the cutting plane and the cone's vertex. The sphere is to touch the side of the cone and will touch the cutting plane at one point, the focus. The directrix is found by creating a plane at the level where the sphere makes contact with the cone, and extending it to intersect with the cutting plane. The line of intersection thus formed is the directrix, Fig. 8.35.

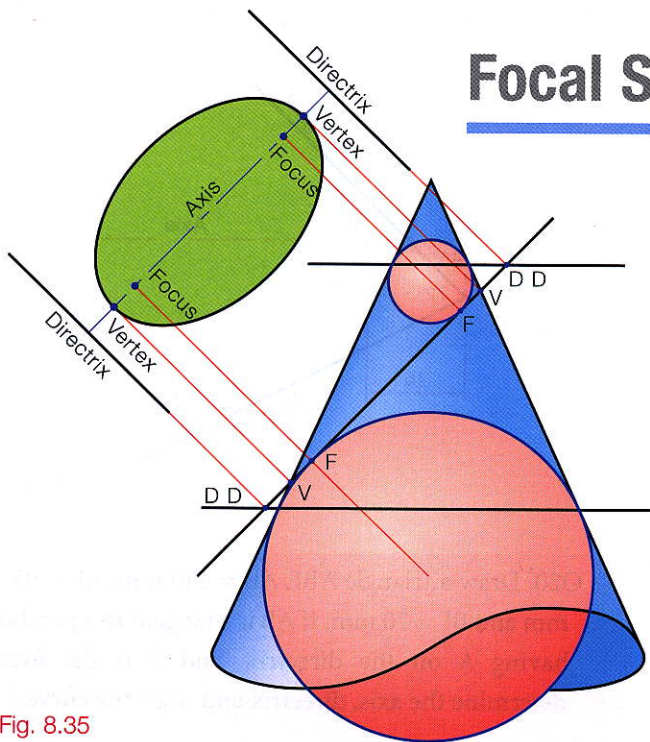


Fig. 8.35

A second focal sphere is found underneath the cutting plane giving the second focus and the second directrix, Fig. 8.36.

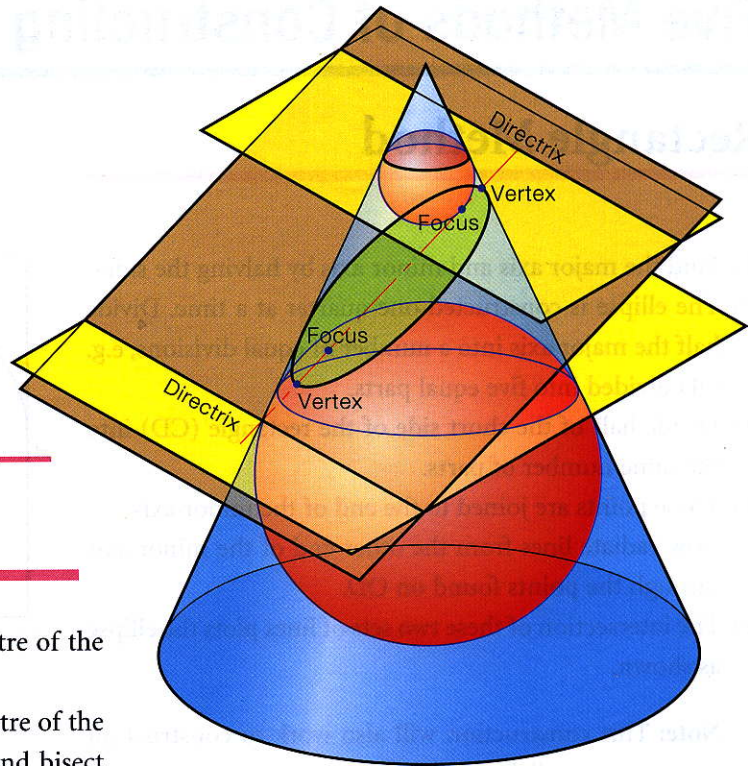


Fig. 8.36

Terminology

Major Axis – The longest axis going through the centre of the ellipse.

Minor Axis – The shortest line going through the centre of the ellipse. The major and minor axes cross at 90° and bisect each other.

Major Auxiliary Circle – The circle passing through both vertices and having its centre on the axis at C.

Minor Auxiliary Circle – The circle passing through both ends of the minor axis and having its centre on the axis at C.

Focal Points – The focal points are two points on the major axis. They are symmetrical about C. They are located a distance of $\frac{1}{2}$ major axis from the ends of the minor axis. Ellipses with focal points near the centre C will be very circular. Ellipses having focal points near the ends of the major axis will be flat ellipses.

Point P – For any point P on the curve the distances F_1P and F_2P added together will equal the length of the major axis.
 $F_1P + F_2P = V_1V_2$ (Major Axis).

A vertical from the focus to hit the major auxiliary circle and then brought across parallel to the major axis will give the top of the minor axis.

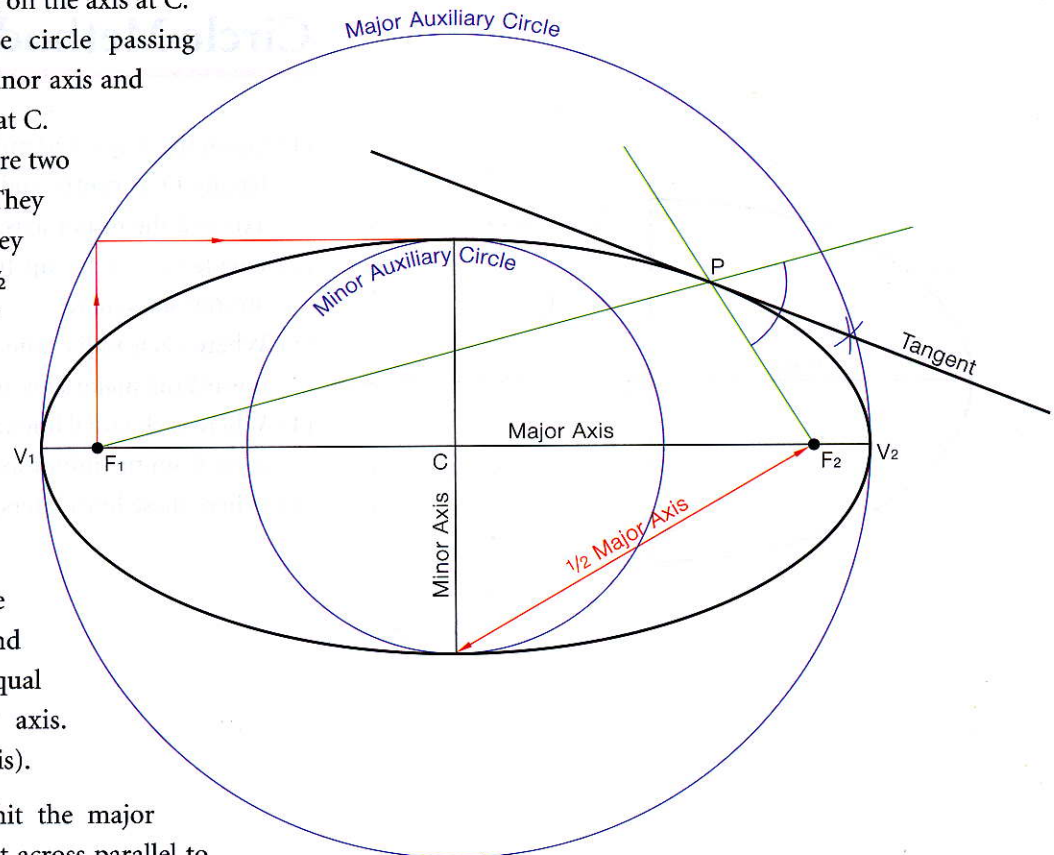


Fig. 8.37

Five Methods of Constructing an Ellipse

Rectangle Method

- (1) Find the major axis and minor axis by halving the sides.
- (2) The ellipse is constructed one quarter at a time. Divide half the major axis into a number of equal divisions, e.g. OD divided into five equal parts.
- (3) Divide half of the short side of the rectangle (CD) into the same number of parts.
- (4) These points are joined to the end of the minor axis.
- (5) Now radiate lines from the other end of the minor axis through the points found on OD.
- (6) The intersection of these two sets of lines plots the ellipse as shown.

Note: This construction will also work to construct an ellipse in a parallelogram.

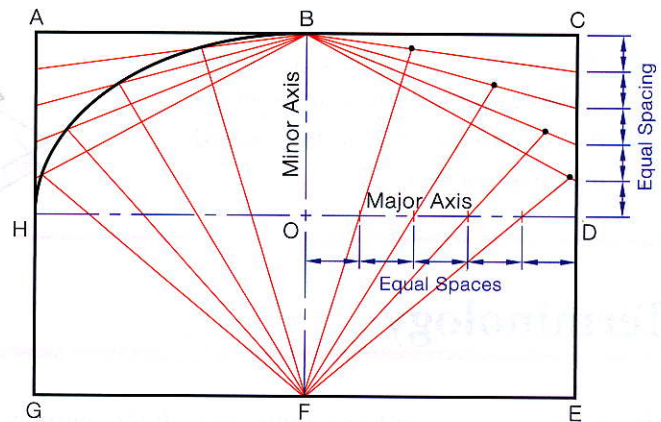


Fig. 8.38

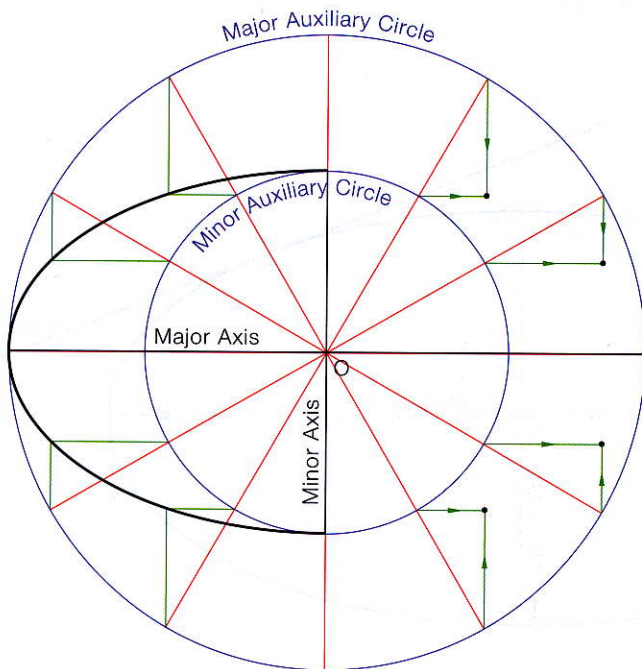


Fig. 8.39

Circle Method

- (1) Given the major and minor axes, construct the two circles having O as centre and radii equal to half of the major axis and the minor axis.
- (2) Divide the circles up from the centre, usually with the 30°/60° set-square.
- (3) Where each radial line hits the major circle, draw lines toward the major axis and parallel to the minor axis.
- (4) Where each radial line crosses the minor circle, draw lines away from the minor axis and parallel to the major axis.
- (5) Where these lines intersect plots points on the curve.

Eccentricity Method

The eccentricity for an ellipse is always **less than 1**, e.g. $3/4$, $2/3$, 0.7 , 0.61 etc. but is a constant for that particular ellipse.

As in the parabola, it is an expression of the relationship between the two distances:

- (i) From the focal point to a point P on the curve.
- (ii) From the same point on the curve to the directrix.

$$\begin{aligned} \text{Eccentricity} &= \frac{\text{Distance from focus to a point}}{\text{Point to directrix}} \\ &= \frac{F \text{ to } P}{P \text{ to } DD} \end{aligned}$$

The eccentricity line for an ellipse will always be at an angle of less than 45° to the axis.

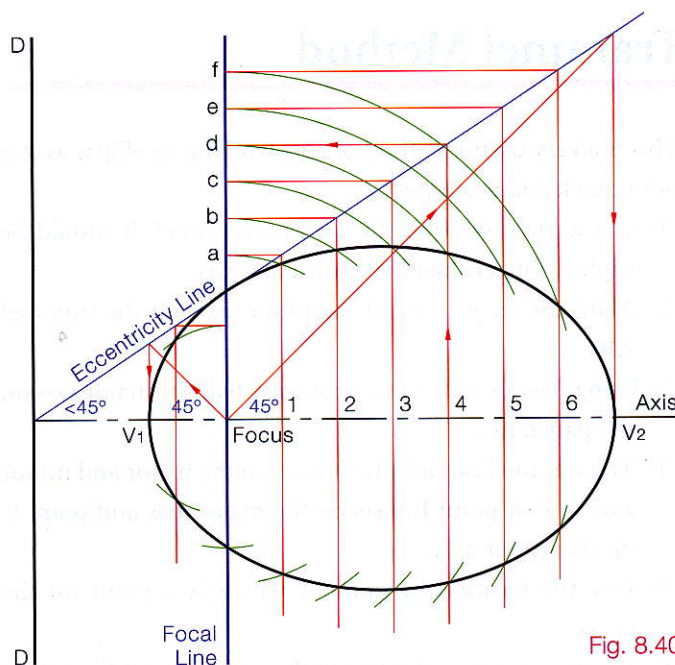


Fig. 8.40

CONSTRUCTION

Given the directrix, axis, focal point and eccentricity of $2/3$.

- (1) Set up the eccentricity line by measuring out on the axis a set distance from the directrix, e.g. 30 mm. Construct a perpendicular to the axis at this point. Measure up from the axis on this line a distance equal to $2/3$ of the previous distance. 20 mm in this example.
- (2) This gives a point on the eccentricity line which now can be drawn.
- (3) Now follow the normal procedure as explained for the parabola in Fig. 8.7.

- (4) The ellipse has two vertices and these are found by constructing 45° lines to the axis from the focus to hit the eccentricity line. These points are projected to give V_1 and V_2 .
- (5) Points on the curve are found by drawing ordinates. Where these intersect the eccentricity line projects across, parallel to the axis, to the focal line.
- (6) With the focus as centre, rotate the points found on the focal line back to each ordinate, above and below the axis.
- (7) The points found in this way can be joined giving an ellipse.

Trammel Method

This is a very useful method of constructing an ellipse as it is both quick and accurate.

- (1) Cut a piece of paper to use as a trammel. It should be slightly longer than half the major axis.
- (2) Mark the length of half the minor axis on the trammel AB.
- (3) Using A as an end point, now mark half the major axis on the paper, AC.
- (4) The trammel can now be placed on the major and minor axis so that point B rests on the major axis and point C on the minor axis.
- (5) Plot the location of point A which is a point on the curve.
- (6) Rotate the trammel, keeping the points B and C on their appropriate axes. Continue to plot points at A.
- (7) Join these plotted points to form an ellipse.

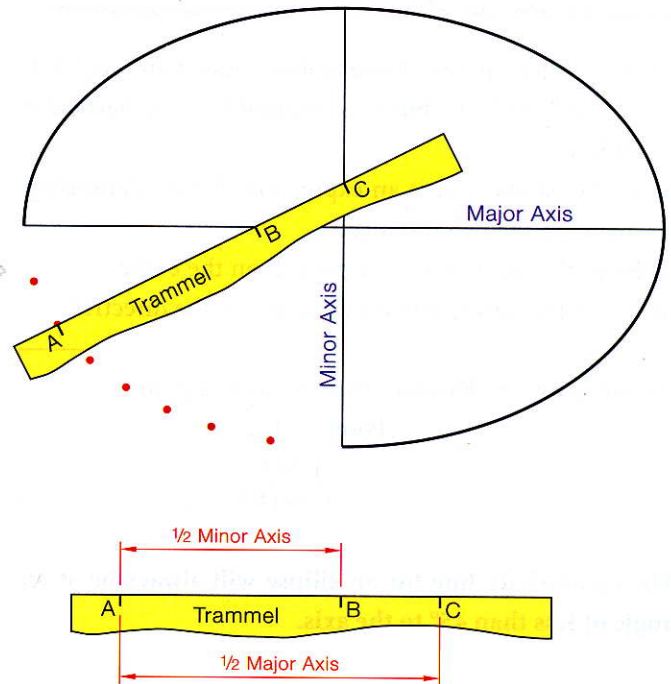


Fig. 8.41

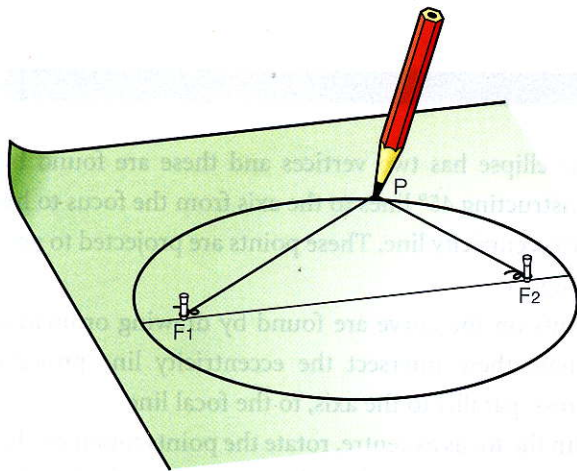


Fig. 8.42

Pin and String Method

This is a good method for drawing large scale ellipses. The method of construction is based on the fact that the distance from F_1P added to F_2P will remain constant and equals the length of the major axis.

$$F_1P + F_2P = \text{Major Axis}$$

- (1) The pins are fixed at the focal points.
- (2) String is tied to the pins so that the amount of string left between the pins equals the length of the major axis of the required ellipse.
- (3) By keeping the string stretched with a pencil and moving it around, an ellipse will be plotted.

Tangent to an Ellipse from a Point on the Curve

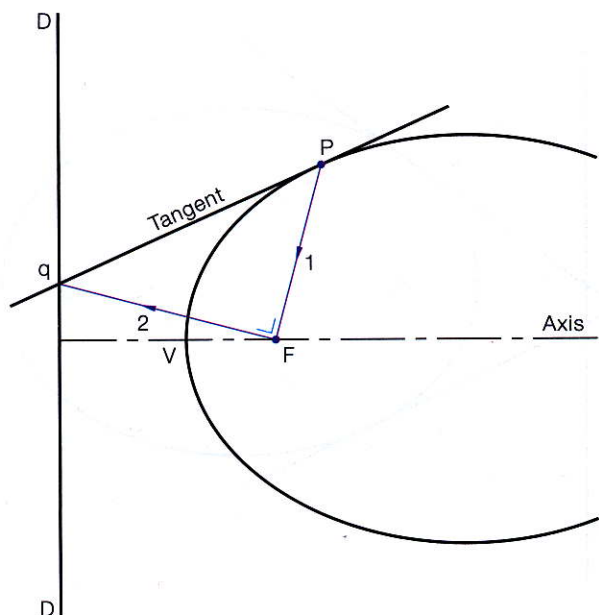


Fig. 8.43

Method 1

Fig. 8.43

This method works for all conics and has already been shown in Fig. 8.8 for the parabola.

- (1) Draw a line joining point P back to the focus.
- (2) At the focus, draw a new line perpendicular to PF and extend to hit the directrix DD at q .
- (3) Point q is a second point on the tangent.
- (4) Join P to q , forming the tangent.

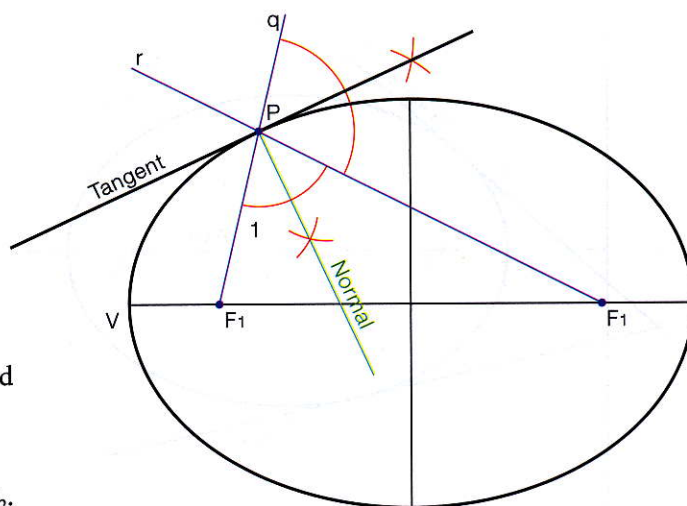


Fig. 8.44

Method 2

Fig. 8.44

- (1) Join F_1 to P and extend to q .
- (2) Join F_2 to P and extend to r .
- (3) The line that bisects the angle F_2Pq will be the required tangent.
- (4) Alternatively F_1Pr could be bisected.
- (5) The normal could be found by bisecting the angle F_1PF_2 . See the similarity to that used for the parabola in Method 3, Fig. 8.10.

Tangent to an Ellipse from a Point P outside the Curve

To draw a tangent to an ellipse from a point P outside the ellipse. Fig 8.45

Method 1

Fig. 8.45

With P as centre and PF_1 as radius, swing an arc to hit the directrix at R and q. Join from R and q back to F_2 giving the points of contact. Draw the tangents.

Note the similarity with construction for parabola, Fig. 8.13.

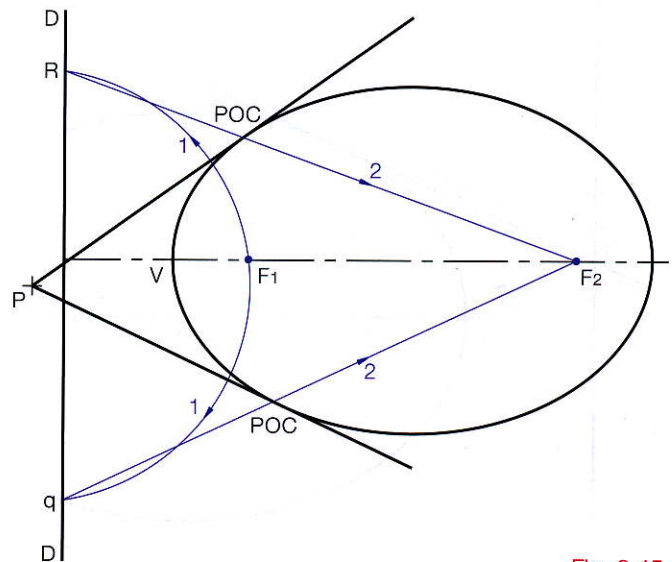


Fig. 8.45

Method 2

Fig. 8.46

Join P to F_1 and place a circle on this line having PF_1 as its diameter. Draw the major auxiliary circle for the ellipse, i.e. the circle which has V_1V_2 as a diameter. These two circles intersect at points r and q, which will be points on the tangents. Draw the tangents.

Note the similarity to construction for a parabola, Fig. 8.14.

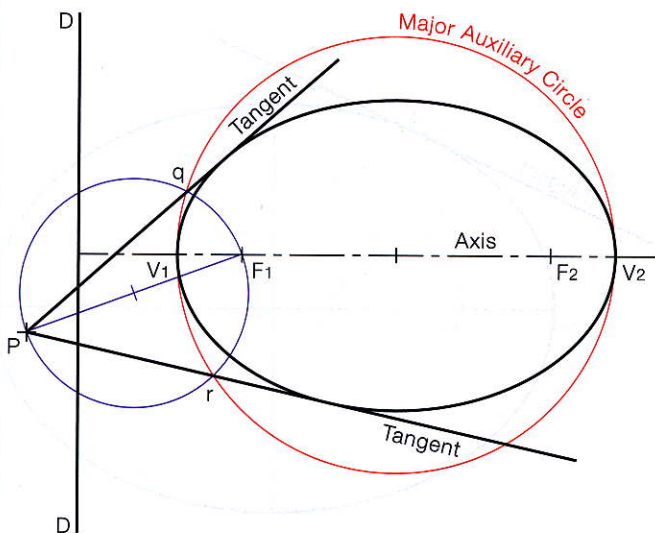


Fig. 8.46

Method 3

Fig. 8.47

With point P as centre and PF_1 as radius, draw an arc. Now take the length of the major axis as radius V_1V_2 . Using F_2 as the centre point swing an arc to cut your first arc in two places, q and r. Points q and r are joined back to F_2 . Where these lines cross the ellipse give the points of contact for the tangents. Draw the tangents.

Note the similarity between the construction for an ellipse and for a hyperbola.

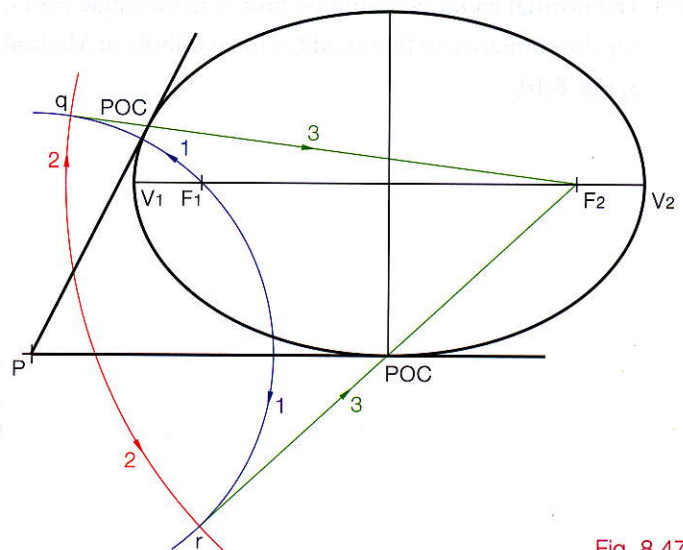
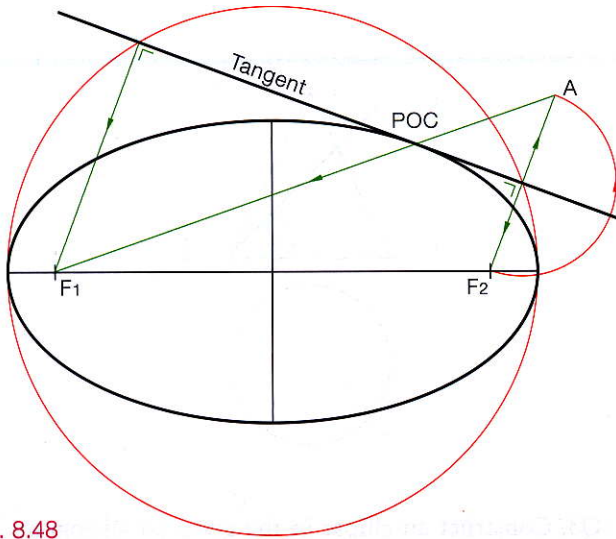


Fig. 8.47



Given a tangent to an ellipse to find the point of contact (POC). Fig 8.48

Fig. 8.48

- (1) Draw the major auxiliary circle.
- (2) Where the tangent and auxiliary circle intersect draw perpendiculars to the tangent. These perpendiculars will pass through the focal points F_1 and F_2 .
- (3) On either of these perpendiculars you double its length as shown, finding point A.
- (4) Join A back to the focus to find the POC.

Fig. 8.48

Given an ellipse to find its axes.

Fig. 8.49

- (1) Draw any two parallel chords.
- (2) Bisect these chords giving A and B.
- (3) Join A and B.
- (4) Bisect the diameter giving C the ellipse centre.
- (5) With C as centre draw a circle to cut the ellipse in three places.
- (6) Lines joining these points will be parallel to the axes.
- (7) Draw the axes through point C.

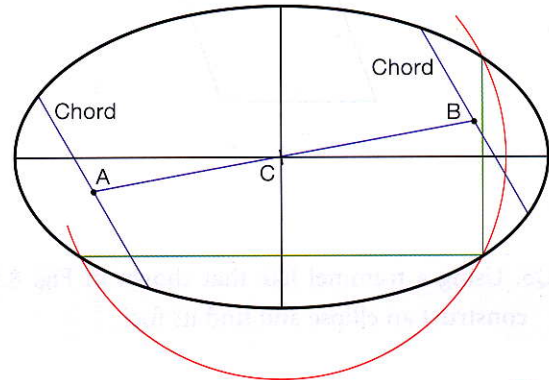


Fig. 8.49

Given a tangent to an ellipse to find its point of contact (POC).

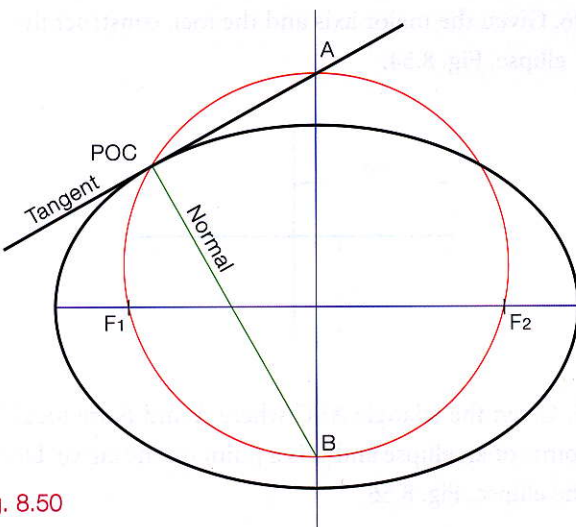


Fig. 8.50

Alternative Method

Fig. 8.50

- (1) Extend the minor axis to intersect the tangent at point A.
- (2) Construct a circle to contain points A, F_1 and F_2 .
- (3) This circle locates the point of contact where it crosses the ellipse.

It should be noted that where the circle intersects the minor axis for the second time at B, it locates a point on the normal. (The angle in a semicircle is always a right angle.)

Activities

Q1. Draw the given plan and elevation of the cone and find the true shape of the section (ellipse), Fig. 8.51.

Q2. Construct an ellipse in a rectangle of side 140 mm by 80 mm using the rectangle method.

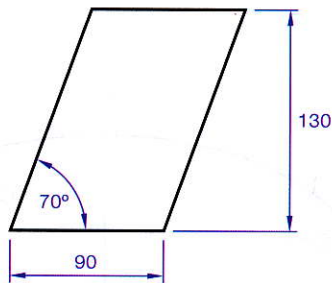


Fig. 8.52

Q5. Using a trammel like that shown in Fig. 8.53, construct an ellipse and find its foci.

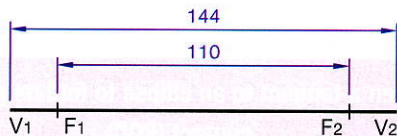


Fig. 8.54

Q7. Given the minor axis and the foci, construct the ellipse, Fig. 8.55.

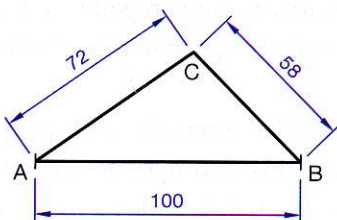


Fig. 8.56

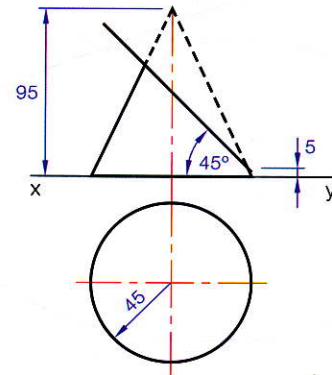


Fig. 8.51

Q3. Construct an ellipse in the given parallelogram such that the sides of the parallelogram form tangents to the ellipse, Fig. 8.52.

Q4. Draw an ellipse having a major axis of 120 mm and a minor axis of 80 mm using the auxiliary circle method. Locate the two focal points.

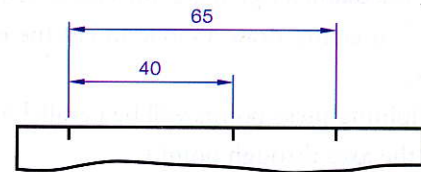


Fig. 8.53

Q6. Given the major axis and the foci, construct the ellipse, Fig. 8.54.

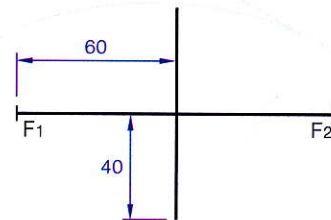


Fig. 8.55

Q8. Given the triangle ABC where A and B are focal points of an ellipse and C is a point on the curve. Draw the ellipse, Fig. 8.56.

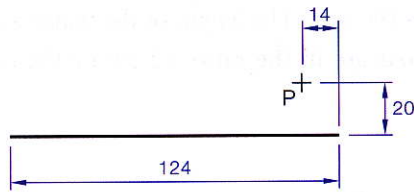


Fig. 8.57

Q10. Given the minor axis of an ellipse and a point P on the curve. Construct the ellipse, Fig. 8.58.

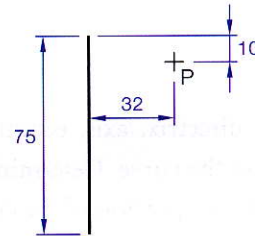


Fig. 8.58

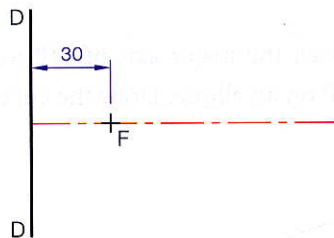


Fig. 8.59

Q12. Given the axis, focus, eccentricity of $3/5$ and a point P on an ellipse. Draw the curve and construct a tangent at point P, Fig. 8.60.

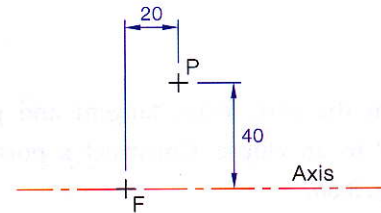


Fig. 8.60

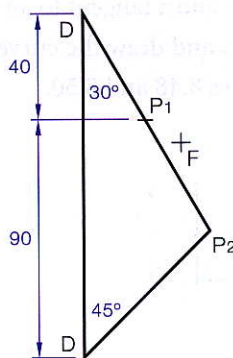


Fig. 8.61

Q13. Construct the triangle shown in Fig. 8.13. DD is the directrix of an ellipse. Points P_1 and P_2 are points on the curve and the eccentricity of the ellipse is $3/4$. F shows the approximate position of the focus. Locate the focus and draw the curve, Fig. 8.61.

$$\text{Eccentricity} = \frac{\text{Distance from focus to a point}}{\text{Point to directrix}}$$

$$= \frac{F \text{ to } P}{P \text{ to } DD}$$

Q14. Given the focus and vertex of an ellipse having an eccentricity of 0.8 . Draw a portion of the curve, Fig. 8.62.

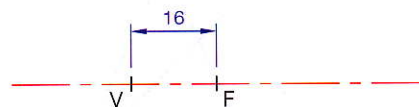


Fig. 8.62

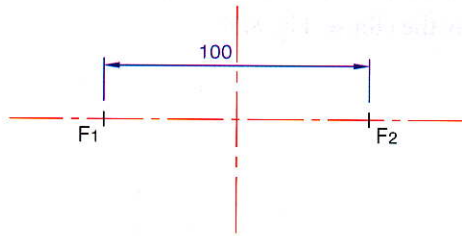


Fig. 8.63

Q16. Given the directrix, axis, eccentricity of 0.75 and a point P on the curve. Determine the position of the focus and draw a portion of the curve. Construct a tangent to the curve from point A, Fig. 8.64.

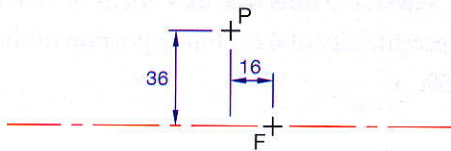


Fig. 8.65

Q18. Given the axis, focus, tangent and point of contact P to an ellipse. Construct a portion of the curve, Fig. 8.66.

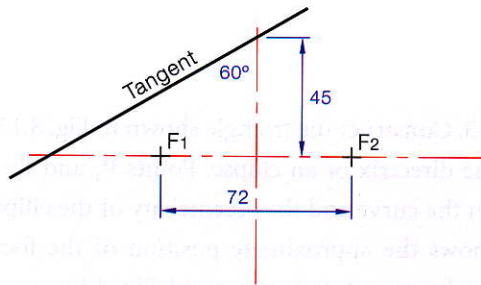


Fig. 8.67

Q20. Given the latus rectum of an ellipse and the vertex. Construct a portion of the curve, Fig. 8.68.

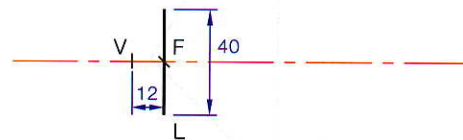


Fig. 8.68

Q15. In an ellipse the distance between the focal points is 100 mm. The length of the major axis and the minor axis are in the ratio 3:2. Draw the ellipse, Fig. 8.63.

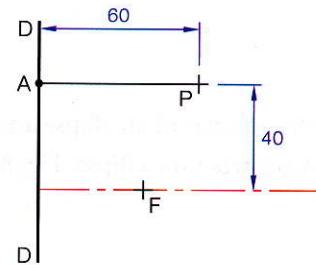


Fig. 8.64

Q17. Given the major axis of 110 mm, focus and a point P on an ellipse. Draw the curve, Fig. 8.65.

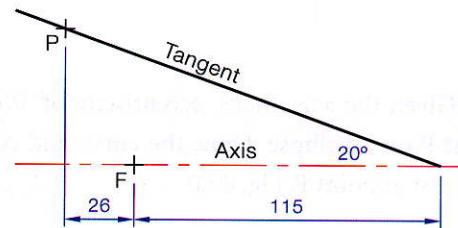


Fig. 8.66

Q19. Given F_1 , F_2 and a tangent to an ellipse. Find the point of contact and draw the curve, Fig. 8.67.

Hint: See Figures 8.48 and 8.50.

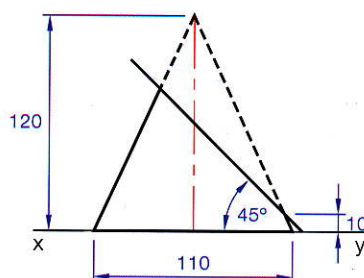


Fig. 8.69

Q21. Given the cone in Fig. 8.69 which is sectioned as shown. Construct the focal spheres and hence find the focus, vertex and directrix of the ellipse. Draw the ellipse.

Q22. Given the triangle APF . AP is a tangent to an ellipse with P as the point of contact. F is a focal point and the major axis is 120 mm long, Fig. 8.70. Draw the ellipse.

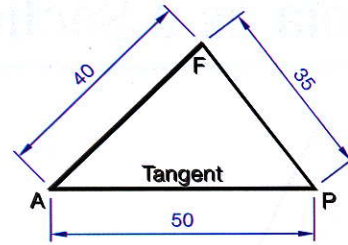


Fig. 8.70

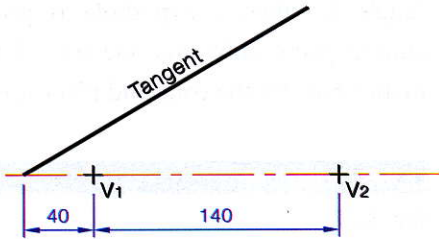


Fig. 8.71

Q24. Given the line P_1FP_2 as shown in Fig. 8.42. F is the focal point of an ellipse and both P_1 and P_2 are points on the curve. The directrix is 50 mm from point P_1 . Draw a portion of the curve. Construct a tangent at P_2 , Fig. 8.72.

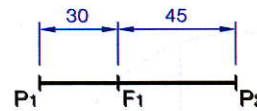


Fig. 8.72

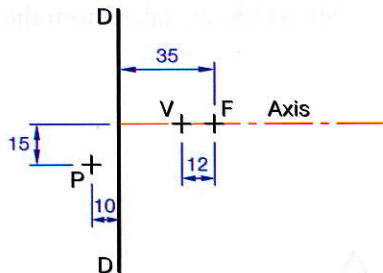


Fig. 8.73

Q26. In Fig. 8.74, AB is a tangent to an ellipse. V is the vertex and F is the focus. Construct the ellipse.

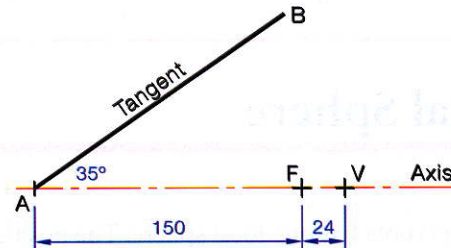


Fig. 8.74

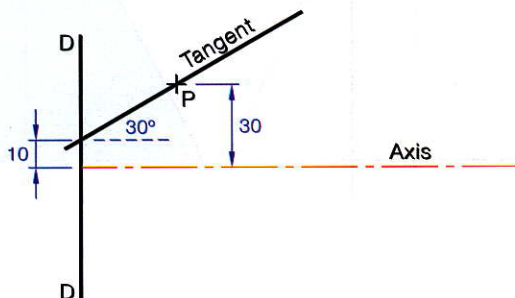


Fig. 8.75

Q28. Given the focus, axis, directrix and tangent to an ellipse. Construct the ellipse and find the point of contact, Fig. 8.76.

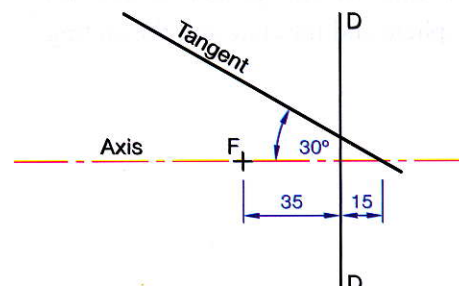


Fig. 8.76